5.4: The Fundamental Theorem of Calculus

Theorem: (Mean Value Theorem) If f is cont. on $[a,b]$ then there exists some $c \in [a,b]$ such that

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

This is the integral analogue to the Intermediate Value theorem.

Fundamental Theorem, Part I: Let f be integrable on $[a,b]$. Define F on $[a,b]$ by

$$F(x) := \int_a^x f(t)\,dt.$$

Then $F(x) =$ "area" under f between $[a,x]$. Notice that just like the derivative we now have a function instead of a value.

The Fund. Thm. of Calc. now tells us that F is differentiable. Furthermore,

$$\frac{F(x+h)-F(x)}{h} = \frac{\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt}{h}$$

$$= \frac{\int_x^{x+h} f(t)\,dt}{h} = \text{Ave Value of } f \text{ on } [x,x+h].$$

But as $h \to 0$, ave value of f on $[x,x+h] \to f(x)$.

Therefore, $\displaystyle\lim_{h\to 0}\frac{F(x+h)-F(x)}{h}=f(x)$. Or

$$F'(x)=f(x).$$

This is the first assertion of the Fund. Thm. of Calc.

<u>FTC (Part I):</u> If $F(x)=\int_a^x f(t)\,dt$, then

$$F'(x)=\frac{d}{dx}\int_a^x f(t)\,dt=f(x).$$

<u>Ex (1):</u> Find $\frac{dy}{dx}$.

(a) $y=\int_a^x t^3+1\,dt \Rightarrow \frac{dy}{dx}=x^3+1$

(b) $y=\int_1^{x^2}\cos t\,dt \Rightarrow y=\int_1^u \cos t\,dt$ where $u=x^2$ so

$$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}=\cos u\cdot 2x=\cos(x^2)\cdot 2x.$$

(c) $y=\int_x^5 3t\sin t\,dt \Rightarrow \frac{dy}{dx}=-3x\sin x.$

(d) $y=\int_{1+3x^2}^4 \frac{1}{2+e^t}\,dt=-\int_4^{1+3x^2}\frac{1}{2+e^t}\,dt \Rightarrow y=-\int_4^u \frac{1}{2+e^t}\,dt$ where $u=1+3x^2$;

so $$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}=-\frac{1}{2+e^u}\cdot 6x$$

$$=-\frac{6x}{2+e^{1+3x^2}}.$$

Fund. Thm. of Calc. (Part II)

$$\int_a^b f(x)dx = F(b) - F(a)$$

for any antiderivative F of f.

In other words, ~~t~~

$$\text{Net Change of } F \text{ on } [a,b] = F(b)-F(a) = \int_a^b F'(x)dx$$

Remark: The Fund. Thm. of Calc. (FTC) asserts that differentiation and integration are opposites.

Ex (2): Evaluate the following.

(a) $\int_0^{\pi} \cos x\,dx = \sin x \Big|_0^{\pi} = \sin(\pi) - \sin(0) = 0$

(b) $\int_{-\pi/4}^{0} \sec x \tan x\,dx = \sec x \Big|_{-\pi/4}^{0} = \sec 0 - \sec(-\pi/4) = 1 - \sqrt{2}.$

(c) $\int_1^4 \frac{3}{2}\sqrt{x} - \frac{4}{x^2}\,dx = x^{3/2} + \frac{4}{x} \Big|_1^4 = \left(4^{3/2} + \frac{4}{4}\right) - \left(1^{3/2} - \frac{4}{1}\right) = 4$

Ex (3): (a) Let $c(x)$ be the cost of producing x units of a good. Then

$$\int_{x_1}^{x_2} c'(x)dx = c(x_2) - c(x_1) = \text{cost of increasing production from } x_1 \text{ units to } x_2 \text{ units.}$$

(b) If $s(t)$ gives an objects position, then

$$\int_{t_1}^{t_2} v(t)dt = s(t_2) - s(t_1) = \text{displacement over } [t_1, t_2].$$

(c) Similarly $\int_{t_1}^{t_2} |v(t)|dt = \text{Total distance traveled}$

<u>Ex(4)</u>: A rock is blown straight up with velocity fcn $v(t)=160-32t$ ft/sec

(a) Displacement on $0 \le t \le 8$ is

$$\int_0^8 160-32t\,dt = 160t-16t^2\Big|_0^8 = (160(8)-16(8)^2)-0$$
$$= 256 \text{ ft.}$$

(b) Total Distance on $0 \le t \le 8$ is

$$\int_0^8 |v(t)|\,dt = \int_0^5 160-32t\,dt - \int_5^8 160-32t\,dt = 544 \text{ ft.}$$

<u>Ex(5)</u>: Find the total Area (actual area) Between $f(x)=x^3-x^2-2x$ and the x-axis on $-1 \le x \le 2$.

$$\int_{-1}^2 |f(x)|\,dx = \int_{-1}^0 x^3-x^2-2x\,dx - \int_0^2 x^3-x^2-2x\,dx = \frac{37}{12}.$$

<u>Remark</u>: The FTC can be restated for initial value problems as

$$F(x) = F(a) + \int_a^x f(x)\,dx.$$